The Gromoll Filtration With Coefficients

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When doing basic things with the Gromoll filtration we have the following useful facts that are indispensible for any analysis:

- The concordance obstructs one step, the blocked diff obstructs many steps
- The HSS connects the two
- Fibrations give LES of homotopy groups
- These fibrations can be arranged in a nice diagram
- The π_0's are related to Θ_n

What we now need to do is establish the anologue of these facts when we take homotopy groups with coefficients.

1 Homotopy Preliminaries

First following Hatcher we denote $\mu_n(X;G) := [M(G,n),X]$ and $\pi_n(X;G) := \mu_{n-1}(X;G)$. Note that only μ_2 and above are groups while μ_3 and above are abelian groups. Moreover only the μ_2 and above define unique homotopy types. To see that μ_1 is not well defined consider Hatcher Ex.2.38 and wedge with an acyclic space with non-trivial fundamental group.

Lemma.

$$\mu_i(BX) = \mu_{i-1}(X) = \mu_{i-2}(\Omega X)$$

This justifies the following, if we assume further that $X = \Omega^n Y$ is an n-fold loop space then we will make the definition $\mu_{2-k}(X) := \mu_2(B^k X)$ for $k \geq 1$. We see that these are groups and well defined up to homotopy. In particular $\mu_0(X) = \mu_2(B^2 X)$ and $\mu_1(X) = \mu_2(BX)$.

Lemma. $\pi_0(BG) = 0$

> ***Proof.*** The total space $EG \to BG$ is contractable and in particular path connected, the continuous image of path connected is path connected and the map is a surjection hence BG is path connected.

2 Comparison Maps

Universal Coefficients (Compares to the tensor) If X is a topological group then applying universal coefficients to BX we get the exact sequence for $* \geq 1$

$$0 \to \pi_*(X) \otimes G \to \mu_*(X; G) \to \mathrm{Tor}(G, \pi_{*+1}(X)) \to 0$$

This provides an explicit comparison map between the homotopy groups with coefficients and the tensored groups.

Puppe Sequence (For $G = \mathbb{Z}_p$) On the other hand if we consider the Puppe sequence for $S^1 \hookrightarrow S^1 \cup_p D^2 = M(\mathbb{Z}_p, 1)$ then we get an exact sequence of the form

$$\cdots \to \pi_2(X) \to \mu_1(X; \mathbb{Z}_p) \to \pi_1(X)$$

Note that if X is a group then we get that the final map here is actually a group homomorphism, however it always exists and is a morphism of pointed spaces. Is that true.

Functoriality We also have from [Nei10, Cor 6.6] that for if we fix a space the homotopy groups with coefficients becomes a functor from finitely generated groups with no two torsion to sets (or groups) for $n \geq 4$. Thus if X is a group we can conclude that

$$\mu_*(X; -) : \text{Groups with no two torsion} \to \text{Group}$$

is a functor for $* \geq 2$. In particular we have the quotient map $\mathbb{Z} \to \mathbb{Z}_p$ for p an odd prime which by functoriality induces a map

$$\pi_{*+1}(X) = \mu_*(X; \mathbb{Z}) \to \mu_*(X; \mathbb{Z}_p)$$

Summary Thus we have the following maps

$$\begin{tikzcd}
\pi_i X \arrow[dr, "\text{functoriality}"] & & \pi_{i-1}X \\
& \pi_i(X; \mathbb{Z}_p) \arrow[ur, "\text{Puppe}"] \arrow[dr, "\text{u.c.t}"] & \\
\pi_{i-1}(X) \otimes \mathbb{Z}_p \arrow[ur, "\text{u.c.t}"] & & \pi_i(X) \otimes \mathbb{Z}_p
\end{tikzcd}$$

which are well defined group homomorphisms for $i \geq 3$. By taking enough deloopings we can make all of this work even on $\pi_0(\mathrm{Diff}_\partial(D^n); G)$, for say $n > 4$. Notice that we can iterate the functoriality and Puppe sequence by connecting the two ends. What is the homology of that sequence? Im very curious what the composition of the map from the top left to bottom right gives, is it zero or tensoring or neither...

2.1 The Base Case

From now on assume that G is a *finite* abelian group. Because we are trying to define a Gromoll filtration one should first ask what does it make sense to filter in this setting. In the normal setting we use that $\pi_0\mathrm{Diff}_\partial(D^{n-1}) \cong \Theta_n$ to filter Θ_n. By smoothing theory we have that $\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}PL_n/\mathcal{O}_n$ we can see that for $n \geq 2$ our space of diffeomorphisms is a triple loop space. In fact for $n = 1, 2, 3$ the space is contractable and therefore by universal coefficients all the Moore space homotopy groups are zero as well. Hence we can always assume that our diffeomorphism group is a triple loop space and we have a well defined group $\pi_0(\mathrm{Diff}_\partial(D^n); G)$, for $n > 4$ where we are interested.

So at least the symbols have meaning now. We can say that we are trying to filter $\pi_0(\mathrm{Diff}_\partial(D^n); G)$, how does this relate to $\pi_0\mathrm{Diff}_\partial(D^n)$. Well by the universal coefficients theorem there is a ses

$$0 \to \pi_{-1}(\mathrm{Diff}_\partial(D^n)) \otimes G \to \pi_0(\mathrm{Diff}_\partial(D^n); G) \to G \otimes \pi_0\mathrm{Diff}_\partial(D^n) \to 0$$

which in more sane notation simply means

$$0 \to \pi_0(B\mathrm{Diff}_\partial(D^n)) \otimes G = 0 \to \pi_0(\mathrm{Diff}_\partial(D^n); G) \to G \otimes \Theta_{n+1} \to 0$$

In $\pi_0(\mathrm{Diff}_\partial(D^n); G) \cong G \otimes \Theta_{n+1}$. The other maps seem less natural, for instance functoriality gives a map $\Theta_{n+1} \to \pi_0(\mathrm{Diff}_\partial(D^n); G)$ and Puppe would give us a map $\pi_1(\mathrm{Diff}_\partial(D^n); G) \to \Theta_{n+1}$, this seems viable to give a filtration of Θ_{n+1} however. If the Cerf with coefficients is true then maybe there is a relation between these two. This map on its own also is very interesting, what is its kernel, what subgroup of Θ does it describe.

3 Defining the filtration

We will define several maps $\pi_i(\mathrm{Diff}_\partial(D^n); \mathbb{Z}_p) \to \pi_{i-1}(\mathrm{Diff}_\partial(D^{n+1}); \mathbb{Z}_p)$ and we will want to show that they are in the end the same map. Recall that by [Nei10, §4] we have long exact sequences for fibrations. **Warning** that his convention for the $\pi_1(-; G)$ differs from mine (his is probably better since it actually fits in the LES).

Assembly Consider the assembly map

$$\Omega\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}_\partial(D^{n+1})$$

given by (carefully) going throught the maps

$$[S^1, [D^n, D^n]] = [D^1, S^0; [D^n, D^n]] \subseteq [D^1 \times D^n, D^n] \subseteq [D^{n+1}, D^{n+1}].$$

Then we claim that the Gromoll map is given by

$$[S^i, [D^n, D^n]] = [\Sigma S^{i-1}, [D^n, D^n]] = [S^{i-1}, \Omega[D^n, D^n]] \xrightarrow{[S^{i-1}, \text{assemble}]} [S^{i-1}, [D^{n+1}, D^{n+1}]]$$

Then this definition generalises straightforwardly to the Moore space homotopy groups since

$$[M(G, n), X] = [\Sigma M(G, n-1), X] = [M(G, n-1), \Omega X].$$

Concordancing We still have the fibration

$$\mathrm{Diff}_\partial(D^{n+1}) \to \mathscr{C}(D^n) \to \mathrm{Diff}_\partial(D^n)$$

which induces a LES on homotopy groups. The boundary map is therefore a map $\pi_*(\mathrm{Diff}_\partial(D^n); G) \to \pi_{*-1}(\mathrm{Diff}_\partial(D^{n+1}; G)$ which we will want to agree with the above assembly map.

Unblocking We still have the fibration

$$\mathrm{Diff}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$$

which induces a LES on homotopy groups. Now we claim two things

- $\pi_i(\widetilde{\mathrm{Diff}}_\partial(D^n);G) \cong \pi_0(\widetilde{\mathrm{Diff}}_\partial(D^{n+i});G) \cong \pi_0(\mathrm{Diff}_\partial(D^{n+i});G) \cong \Theta_{n+i+1} \otimes G$
- The image of the map $\pi_i(\mathrm{Diff}_\partial(D^n);G) \to \pi_i(\widetilde{\mathrm{Diff}}_\partial(D^n);G)$ is the image of the itterated assembly map in $\pi_0(\mathrm{Diff}_\partial(D^{n+i});G)$

3.1 Summary

For p an odd prime we have the situtation

$$\begin{tikzcd}
\pi_i\mathrm{Diff}_\partial(D^n) \arrow[r, "\text{functoriality}"'] \arrow[d] & \pi_i(\mathrm{Diff}_\partial(D^n);\mathbb{Z}_p) \arrow[r, "\text{Puppe}"'] \arrow[d] & \pi_{i-1}\mathrm{Diff}_\partial(D^n) \arrow[d] \\
\pi_{i-1}\mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "\text{functoriality}"'] & \pi_{i-1}(\mathrm{Diff}_\partial(D^{n+1});\mathbb{Z}_p) \arrow[r, "\text{Puppe}"] & \pi_{i-2}\mathrm{Diff}_\partial(D^{n+1})
\end{tikzcd}$$

Where all the verticle maps are assembly. Does this diagram commute, what can we say about the exactness of the horrozontals. We can continue the series to the left and right by itterating it, possibly stiching in Puppe maps etc, does this have any meaning, or application? Its again interlinking long sequences can I get a SS from this?

4 Blocked Proofs

Recall from smoothing theory [BL74] we have the fibration

$$PL_n/O_n \to PL/O \to C$$

which we can loop $n+1$ times to obtain

$$\mathrm{Diff}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$$

Notice then that we can deloop both the fibration and the fiber.

Lemma.

$$\pi_0(\widetilde{\mathrm{Diff}}_\partial(D^{n+i});G) \cong \pi_0(\mathrm{Diff}_\partial(D^{n+i});G)$$

Proof. Deloop the fibration 5 or so times to be safe. Then the LES in homotopy groups with coefficients says

$$\cdots \to \pi_6(\Omega^{n+1-5}C;G) \to \pi_5(\Omega^{n+1-5}PL_n/O_n;G) \to \pi_5(\Omega^{n+1-5}PL/O;G) \to \pi_5(\Omega^{n+1-5}C;G) \to \cdots$$

Now we claim that $\pi_6(\Omega^{n+1-5}C;G) = \pi_5(\Omega^{n+1-5}C;G) = 0$. Applying universal coefficients gives us the SES's

$$0 \to \pi_4(\Omega^{n+1-5}C) \otimes G \to \pi_5(\Omega^{n+1-5}C;G) \to \pi_5(\Omega^{n+1-5}C) \otimes G \to 0$$

$$0 \to \pi_5(\Omega^{n+1-5}C) \otimes G \to \pi_6(\Omega^{n+1-5}C;G) \to \pi_6(\Omega^{n+1-5}C) \otimes G \to 0$$

So it is clearly sufficient for $\pi_4(\Omega^{n+1-5}C) = \pi_5(\Omega^{n+1-5}C) = \pi_6(\Omega^{n+1-5}C) = 0$.

$$\pi_4(\Omega^{n+1-5}C) = \pi_0\Omega^n C = \pi_0 B\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n) = 0$$

$$\pi_5(\Omega^{n+1-5}C) = \pi_0\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n) = 0$$

$$\pi_6\Omega^{n+1-5}C = \pi_1\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n) = 0$$

Hence in the LES of the fibration we have

$$\cdots \to 0 \to \pi_5(\Omega^{n+1-5}PL_n/O_n;G) \to \pi_5(\Omega^{n+1-5}PL/O;G) \to 0 \to \cdots$$

And we see that

$$\pi_0(\mathrm{Diff}_\partial(D^n);G) = \pi_5(\Omega^{n+1-5}PL_n/O_n;G) \to \pi_5(\Omega^{n+1-5}PL/O;G) = \pi_0(\widetilde{\mathrm{Diff}}_\partial(D^n);G).$$

□

Lemma.

$$\pi_i(\widetilde{\mathrm{Diff}}_\partial(D^n);G) \cong \pi_{i-1}(\widetilde{\mathrm{Diff}}_\partial(D^n);G)$$

Proof. First we know from smoothing theory [BL74, Thm 4.5] that on the level of spaces we have

$$\widetilde{\mathrm{Diff}}_\partial(D^n) \simeq \Omega^{n+1}PL/O$$

which we can use to see that

$$\pi_i\widetilde{\mathrm{Diff}}_\partial(D^n) \cong \pi_i\Omega^{n+1}PL/O \cong \pi_{i+1}\Omega^n PL/O \cong \pi_{i+1}\widetilde{\mathrm{Diff}}_\partial(D^{n-1})$$

This uses only that $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ and so if that is true with coefficients we know at least abstaractly that with coefficients the groups are all isomorphic still. Moreover one would hope by functoriality and commutativity of a diagram on the level of spaces we get what we want. Elaborated on in the other doc

Lemma. *The following diagram commutes*

$$\begin{tikzcd}
\pi_i(\mathrm{Diff}_\partial(D^n);G) \arrow[r] \arrow[d, "assemble^i"] & \pi_i(\widetilde{\mathrm{Diff}}_\partial(D^n);G) \arrow[r] & \pi_i(\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n);G) \\
\pi_0(\mathrm{Diff}_\partial(D^{n+i-1});G) \arrow[r, "\cong"] & \pi_0(\widetilde{\mathrm{Diff}}_\partial(D^{n+i-1});G) \arrow[u, "{assemble^{-i},\cong}"] &
\end{tikzcd}$$

Proof. The normal proof is [ABK72, 2.3.3]; not immediate that it generalises, to be honest I can even understand their proof.

Remark. Want to prove that the map induced on homotopy groups is the same either from the concordance LES or the assembly map, that is reconstruct the concordance Gromoll picture.

Remark. [Hatcher Spectral Sequence] See how much we still get from the Hatcher spectral sequence. Notice that this seuqence is two torsion, does that tell me anything. Can I still use the geometric description of the maps, can I compute the blocked spaces homotopy groups with coefficients, can I use Weiss-Williams still.

Remark. Can we prove the Cerf theorem for homotopy groups with coefficients, shouldnt be too bad (probably follows from above).

References

[ABK72] P. L. Antonelli, D. Burghelea, and P. J. Kahn. The non-finite homotopy type of some diffeomorphism groups. *Topology*, 11(1):1–49, January 1972.

[BL74] Dan Burghelea and Richard Lashof. The homotopy type of the space of diffeomorphisms. II. *Transactions of the American Mathematical Society*, 196(0):37–50, 1974.

[Nei10] Joseph A. Neisendorfer. Homotopy groups with coefficients. *Journal of Fixed Point Theory and Applications*, 8(2):247–338, December 2010.